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On the co-ordinated $g-$convex dominated functions

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Abstract

In this study, we define $g-$convex dominated functions on the co-ordinates and prove some Hadamard-type, Fejer-type inequalities for this new class of functions. We also give some results related to the functional H.

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1 Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0,1]$. The classical Hermite-Hadamard inequality gives us an estimate of the mean value of a convex function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ which is well-known in the literature as following;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

In [8], Dragomir defined convex functions on the co-ordinates as following;

Definition 1. Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$. Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex on Δ if the following inequality holds,

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w)$$

for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

Every convex function is co-ordinated convex but the converse is not generally true.

In [8], Dragomir established the following inequalities of Hadamard's type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

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Theorem 1. Suppose that $f:\Delta=[a,b]\times[c,d]\rightarrow\mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities;

$$\begin{aligned}
& f(\frac{a+b}{2},\frac{c+d}{2}) \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,\frac{c+d}{2})dx+\frac{1}{d-c}\int_c^d f(\frac{a+b}{2},y)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\
\leq\ & \frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x,c)dx+\frac{1}{(b-a)}\int_a^b f(x,d)dx\right. \\
& \left.+\frac{1}{(d-c)}\int_c^d f(a,y)dy+\frac{1}{(d-c)}\int_c^d f(b,y)dy\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

The above inequalities are sharp.

In [11], Alomari and Darus proved following inequalities of Fejer-type for co-ordinated convex functions.

Theorem 2. Let $f:[a,b]\times[c,d]\rightarrow\mathbb{R}$ be a co-ordinated convex function. Then the following double inequality holds:

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{\int_a^b\int_c^d f(x,y)\,p(x,y)\,dydx}{\int_a^b\int_c^d p(x,y)\,dydx} \\
&\leq \frac{f(a,c)+f(c,d)+f(b,c)+f(b,d)}{4}
\end{aligned} \tag{1.1}$$

where $p:[a,b]\times[c,d]\rightarrow\mathbb{R}$ is positive, integrable and symmetric about $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$. The above inequalities are sharp.

In [2], Dragomir *et al.* defined $g-$convex dominated functions and gave some results related to this functions as following;

Definition 2. Let $g:I\rightarrow\mathbb{R}$ be a given convex function on the interval I from $\mathbb{R}$. The real function $f:I\rightarrow\mathbb{R}$ is called $g-$convex dominated on I if the following condition is satisfied:

$$\begin{aligned}
& \left|\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)\right| \\
\leq\ & \lambda g(x)+(1-\lambda)g(y)-g(\lambda x+(1-\lambda)y)
\end{aligned}$$

for all $x,y\in I$ and $\lambda\in[0,1]$. For related Theorems and classes see: [5]-[6]-[7].

Theorem 3. Let $g : I \to \mathbb{R}$ be a convex function on I and $f : I \to \mathbb{R}$. The following statements are equivalent:

(i) f is $g-$convex dominated on I;

(ii) The mappings $g - f$ and $g + f$ are convex on I;

(iii) There exist two convex mappings h, k defined on I such that

$$f = \frac{1}{2}(h - k) \text{ and } g = \frac{1}{2}(h + k).$$

In [8], Dragomir considered a mapping which closely connected with above inequalities and established main properties of this mapping as following:

Now, for a mapping $f : \Delta := [a, b] \times [c, d] \to \mathbb{R}$ is convex on the co-ordinates on Δ, we can define the mapping $H : [0, 1]^2 \to \mathbb{R}$,

$$H(t, s) := \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) dxdy$$

Theorem 4. [See [8]] Suppose that $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is convex on the co-ordinates on $\Delta = [a, b] \times [c, d]$. Then:

(i) The mapping H is convex on the co-ordinates on $[0, 1]^2$.

(ii) We have the bounds

$$\sup_{(t,s)\in[0,1]^2} H(t, s) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)\, dxdy = H(1, 1)$$

$$\inf_{(t,s)\in[0,1]^2} H(t, s) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = H(0, 0)$$

(iii) The mapping H is monotonic nondecreasing on the co-ordinates.

In this study, we define $g-$ convex dominated functions on Δ on the co-ordinates and establish some inequalities of Hadamard-type and Fejer-type for this class of functions. We also give some results for the functional H.

2 Main results

We will start with the following definition.

Definition 3. Let $\Delta = [a, b] \times [c, d]$ and $g : \Delta \to \mathbb{R}$ be a convex function. The real function $f : \Delta \to \mathbb{R}$ is called a $g-$ convex dominated function on Δ if

$$\begin{aligned} &|\lambda f(x, y) + (1-\lambda) f(z, w) - f(\lambda x + (1-\lambda) z, \lambda y + (1-\lambda) w)| \\ \leq\ & \lambda g(x, y) + (1-\lambda) g(z, w) - g(\lambda x + (1-\lambda) z, \lambda y + (1-\lambda) w) \end{aligned}$$

for all $\lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$.

Definition 4. Suppose that $g : \Delta \to \mathbb{R}$ be a convex function. A function $f : \Delta \to \mathbb{R}$ is called $g-$convex dominated on co-ordinates if the partial mappings $f_x : [c,d] \to \mathbb{R}$, $f_x(u) := f(x,v)$ is g_x- convex dominated on $[c,d]$ and $f_y : [a,b] \to \mathbb{R}$, $f_y(w) := f(w,y)$ is g_y- convex dominated on $[a,b]$ where $g_x : [c,d] \to \mathbb{R}$, $g_x(u) := g(x,u)$ and $g_y : [a,b] \to \mathbb{R}$, $g_y(w) := g(w,y)$.

Lemma 1. Let $g : \Delta \to \mathbb{R}$ be a convex function. Every $g-$convex dominated mapping on Δ is $g-$convex dominated on the co-ordinates but the converse may not be necessarily true.

Proof. Suppose that $f : \Delta \to \mathbb{R}$ is $g-$convex dominated on Δ. Consider $f_x : [c,d] \to \mathbb{R}$, $f_x(u) := f(x,u)$ and $g_x : [c,d] \to \mathbb{R}$, $g_x(u) := g(x,u)$. Then for all $\lambda \in [0,1]$ and $u, w \in [c,d]$, we have

$$\begin{aligned} & \left|\lambda f(x,u) + (1-\lambda) f(y,w) - f(\lambda x + (1-\lambda)x, \lambda u + (1-\lambda)w)\right| \\ \leq & \ \lambda g(x,u) + (1-\lambda) g(y,w) - g(\lambda x + (1-\lambda)x, \lambda u + (1-\lambda)w) \end{aligned}$$

then we can write

$$\begin{aligned} = & \ \left|\lambda f(x,u) + (1-\lambda) f(y,w) - f(x, \lambda u + (1-\lambda)w)\right| \\ \leq & \ \lambda g(x,u) + (1-\lambda) g(y,w) - g(x, \lambda u + (1-\lambda)w) \end{aligned} \tag{2.1}$$

it follows that

$$\begin{aligned} & \left|\lambda f_x(u) + (1-\lambda) f_x(w) - f_x(\lambda u + (1-\lambda)w)\right| \\ \leq & \ \lambda g_x(u) + (1-\lambda) g_x(w) - g_x(\lambda u + (1-\lambda)w). \end{aligned} \tag{2.2}$$

We can show that this results also hold for f_y.This completes the proof. Now consider the mappings $g_0(x,y) = x+y$ and $f_0(x,y) = xy$ where $\Delta := [0,1] \times [0,1]$. g_0 is convex on Δ and f_0 is g_0- convex dominated on the co-ordinates but it can be easily seen that the mapping f_0 is not $g_0 - convex$ dominated on Δ. ∎

Lemma 2. Let g be a convex function on Δ and $f : \Delta \to \mathbb{R}$. If f is $g-$convex dominated on the co-ordinates, the mappings $g-f$ and $g+f$ are convex on the co-ordinates.

Proof. From the fact that f_x is g_x-convex dominated we have

$$\begin{aligned} & \left|\lambda g_x(u) + (1-\lambda) g_x(w) - g_x(\lambda u + (1-\lambda)w)\right| \\ \leq & \ \lambda f_x(u) + (1-\lambda) f_x(w) - f_x(\lambda u + (1-\lambda)w). \end{aligned}$$

From Lemma 1, we have $(g-f)_x$ and $(g+f)_x$ are convex on the co-ordinates for all $\lambda \in [0,1]$ and $u, w \in [c,d]$. Similarly we can show that $(g-f)_y$ and $(g+f)_y$ are also convex on the co-ordinates. This completes proof. ∎

Theorem 5. Let $g : \Delta = [a,b] \times [c,d] \to R$ be a co-ordinated convex mapping on Δ and $f : \Delta = [a,b] \times [c,d] \to R$ is a co-ordinated $g-$convex-dominated mapping, where $a < b$ and $c < d$. Then, one has the inequalities:

$$\begin{aligned} & \left|\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dydx - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right| \\ \leq & \ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(x,y)\,dydx - g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \end{aligned}$$

and

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right|$$
$$\leq \quad \frac{g(a,c)+g(a,d)+g(b,c)+g(b,d)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)\,dydx.$$

Proof. From Lemma 2, we can write

$$\left|\frac{f(x,y)+f(z,w)}{2}-f\left(\frac{x+z}{2},\frac{y+w}{2}\right)\right|$$
$$\leq \quad \frac{g(x,y)+g(z,w)}{2}-g\left(\frac{x+z}{2},\frac{y+w}{2}\right)$$

for all $(x,y),(z,w)\in\Delta$. Set $x=ta+(1-t)b$, $z=(1-t)a+tb$, $y=tc+(1-t)d$ and $w=(1-t)c+td$ for all $t,s\in[0,1]$. Then we get

$$\left|\frac{f(ta+(1-t)b,tc+(1-t)d)+f((1-t)a+tb,(1-t)c+td)}{2}-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|$$
$$\leq \quad \frac{g(ta+(1-t)b,tc+(1-t)d)+g((1-t)a+tb,(1-t)c+td)}{2}-g\left(\frac{a+b}{2},\frac{c+d}{2}\right).$$

By integrating with respect to t over $[0,1]^2$, we obtain the desired result.

$$\left|\frac{\int_0^1\int_0^1 f(ta+(1-t)b,tc+(1-t)d)\,dtds}{2}\right.$$
$$\left.+\frac{\int_0^1\int_0^1 f((1-t)a+tb,(1-s)c+sd)\,dtds}{2}-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|$$
$$\leq \quad \frac{\int_0^1\int_0^1 g(ta+(1-t)b,sc+(1-s)d)\,dtds}{2}$$
$$+\frac{\int_0^1\int_0^1 g((1-t)a+tb,(1-s)c+sd)\,dtds}{2}.-g\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

Which completes the proof of the first inequality. From the definition of $g-$convex-dominated functions, we can write

$$\begin{aligned}
& |tsf(a,c)+t(1-s)f(a,d)+s(1-t)f(b,c) \\
& +(1-t)(1-s)f(b,d)-f(ta+(1-t)b,sc+(1-s)d)| \\
\leq\ & tsg(a,c)+t(1-s)g(a,d)+s(1-t)g(b,c) \\
& +(1-t)(1-s)g(b,d)-g(ta+(1-t)b,sc+(1-s)d).
\end{aligned}$$

By integrating with respect to t over $[0,1]^2$, we obtain

$$\begin{aligned}
& \left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right| \\
\leq\ & \frac{g(a,c)+g(a,d)+g(b,c)+g(b,d)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)\,dydx.
\end{aligned}$$

Which completes the proof. ∎

We will prove some properties of the mapping H in the following theorem.

Theorem 6. Let $g:\Delta=[a,b]\times[c,d]\to R$ be a co-ordinated convex mapping on Δ and $f:\Delta=[a,b]\times[c,d]\to R$ is a co-ordinated $g-$convex-dominated mapping, where $a<b$ and $c<d$. Then:

(i) H_f is H_g-convex dominated on $[0,1]$.

(ii) One has the inequalities

$$0\leq|H_f(t_2,s_2)-H_f(t_1,s_1)|\leq H_g(t_2,s_2)-H_g(t_1,s_1) \tag{2.3}$$

for all $0\leq t_2<t_1\leq 1$ and $0\leq s_2<s_1\leq 1$.

(iii) One has the inequalities

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-H_f(t,s)\right|\leq H_g(t,s)-g\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

and

$$\begin{aligned}
& \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-H_f(t,s)\right| \\
\leq\ & H_g(t,s)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)\,dydx
\end{aligned}$$

for all $t,s\in[0,1]$.

Proof. (i) Since f is co-ordinated $g-$convex-dominated mapping on Δ, we know that from Lemma 2, $g-f$ and $g+f$ are also convex on Δ. By using the linearity of the mapping $f\to H_f$, we can write $H_{g-f} = H_g - H_f$ and $H_{g+f} = H_g + H_f$. Then, it is easy to see that H_f is H_g-convex dominated on $[0,1]$.

(ii) By Theorem 4, the mappings H_{g-f} and H_{g+f} are monotonic nondecreasing on the co-ordinates. So, we have

$$H_g(t_1,s_1) - H_f(t_1,s_1) = H_{g-f}(t_1,s_1) \le H_{g-f}(t_2,s_2) = H_g(t_2,s_2) - H_f(t_2,s_2)$$

and

$$H_g(t_1,s_1) + H_f(t_1,s_1) = H_{g+f}(t_1,s_1) \le H_{g+f}(t_2,s_2) = H_g(t_2,s_2) + H_f(t_2,s_2).$$

Then, we obtain

$$H_f(t_2,s_2) - H_f(t_1,s_1) \le H_g(t_2,s_2) - H_g(t_1,s_1)$$

and

$$H_f(t_1,s_1) - H_f(t_2,s_2) \le H_g(t_2,s_2) - H_g(t_1,s_1).$$

Which is the desired result.

(iii) If we choose $t = s = 0$ in (2.3), we have the first inequality and by a similar argument if we choose $t = s = 1$, we obtain the second inequality. ∎

We will establish inequalities of Fejer-type for $g-$convex dominated function on the co-ordinates in the following theorem.

Theorem 7. Let $g : \Delta \to \mathbb{R}$ be a convex function on the co-ordinates and $f : \Delta \to \mathbb{R}$ be a $g-$convex dominated function on the co-ordinates where $\Delta := [a,b] \times [c,d]$. Then the following inequalities hold:

$$\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{\int_a^b\int_c^d f(x,y)p(x,y)\,dydx}{\int_a^b\int_c^d p(x,y)\,dydx} \right| \le g\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{\int_a^b\int_c^d g(x,y)p(x,y)\,dydx}{\int_a^b\int_c^d p(x,y)\,dydx}$$

and

$$\left| \frac{f(a,c)+f(c,d)+f(b,c)+f(b,d)}{4} - \frac{\int_a^b\int_c^d f(x,y)\,p(x,y)\,dydx}{\int_a^b\int_c^d p(x,y)\,dydx} \right| \le \frac{g(a,c)+g(c,d)+g(b,c)+g(b,d)}{4} - \frac{\int_a^b\int_c^d g(x,y)\,p(x,y)\,dydx}{\int_a^b\int_c^d p(x,y)\,dydx}$$

where $p : [a,b] \times [c,d] \to \mathbb{R}$ is positive, integrable and symmetric about $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$.

Proof. Since f is $g-$convex dominated on the co-ordinates, from Lemma 2, the mappings $f+g$ and $g-f$ are convex on the coordinates. By using Theorem 2, we have:

$$(f+g)\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{\int_a^b\int_c^d(f+g)(x,y)\,p(x,y)\,dydx}{\int_a^b\int_c^d p(x,y)\,dydx}$$

and

$$\frac{\int_a^b\int_c^d(f+g)(x,y)\,p(x,y)\,dydx}{\int_a^b\int_c^d p(x,y)\,dydx}\leq\frac{(f+g)(a,c)+(f+g)(c,d)+(f+g)(b,c)+(f+g)(b,d)}{4}.$$

If we re-arrange these inequalities, we obtain:

$$\begin{aligned}&\int_a^b\int_c^d g(x,y)\,p(x,y)\,dydx-g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ \leq\;& f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{\int_a^b\int_c^d f(x,y)p(x,y)\,dydx}{\int_a^b\int_c^d p(x,y)\,dydx}\\ \leq\;& g\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{\int_a^b\int_c^d g(x,y)\,p(x,y)\,dydx}{\int_a^b\int_c^d p(x,y)\,dydx}\end{aligned}$$

and

$$\frac{\int_a^b\int_c^d g(x,y)\,p(x,y)\,dydx}{\int_a^b\int_c^d p(x,y)\,dydx}-\frac{g(a,c)+g(c,d)+g(b,c)+g(b,d)}{4}$$

$$\leq \frac{f(a,c)+f(c,d)+f(b,c)+f(b,d)}{4}-\frac{\int_a^b\int_c^d f(x,y)\,p(x,y)\,dydx}{\int_a^b\int_c^d p(x,y)\,dydx}$$

$$\leq \frac{g(a,c)+g(c,d)+g(b,c)+g(b,d)}{4}-\frac{\int_a^b\int_c^d g(x,y)\,p(x,y)\,dydx}{\int_a^b\int_c^d p(x,y)\,dydx}.$$

Which completes the proof. ∎

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